18 Sturm-Liouville Eigenvalue Problems

Up until now all our eigenvalue problems have been of the form

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \;, \qquad 0 < x < l \tag{1}$$

plus a mix of boundary conditions, generally being Dirichlet or Neumann type. This is too narrow of a viewpoint, which I wish to point out through a few examples.

Example 1: Simple spatial variation in diffusivity: $D = D_0(1+x)^2$.
Consider the problem

$$\begin{cases} u_t = D_0[(1+x)^2 u_x]_x & 0 < x < 1 \;,\; t > 0 \\ u(x,0) = f(x) & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \end{cases} \tag{2}$$

From the separation of variables method, $u(x,t) = T(t)\phi(x)$, we obtain

$$\frac{dT}{dt} = -\lambda D_0 T \;\text{ and }\; \frac{d}{dx}[(1+x)^2\frac{d\phi}{dx}] + \lambda\phi = 0 \;,$$

with $\phi(0) = \phi(1) = 0$. Note that by carrying through the differentiation, the ϕ equation

$$(1+x)^2\frac{d^2\phi}{dx^2} + 2(1+x)\frac{d\phi}{dx} + \lambda\phi = 0 \tag{3}$$

is a *Cauchy-Euler* equation (recall the Review of ODEs appendix for a discussion of the equations). If we write $\phi(x) = (1+x)^r$, the characteristic equation for r becomes

$$r(r-1) + 2r + \lambda = 0 \;\;\Rightarrow\;\; r = \{-1 \pm \sqrt{1-4\lambda}\}/2 \;.$$

Because we have to start at $\phi = 0$ at $x = 0$, and end at $\phi = 0$ at $x = 1$, assume we need oscillatory solutions like we got out of (1), since $\lambda \geq 0$. Hence, assume $\lambda > 1/4$, and define (for notational convenience) $\omega := \sqrt{\lambda - 1/4}$. Then the roots can be written as $r = -1/2 \pm i\omega$, and since

$$(1+x)^{-1/2\pm i\omega} = (1+x)^{-1/2}(1+x)^{\pm i\omega} = (1+x)^{-1/2}e^{\pm i\omega ln(1+x)} \;,$$

a suitable fundamental set of solutions to (3) is

$$(1+x)^{-1/2}\cos(\omega ln(1+x))\ ,\ (1+x)^{-1/2}\sin(\omega ln(1+x))\ .$$

Now $\phi(x)$ can be written as a linear combination of these functions, and since $\phi(0)=0$, we have $\phi(x)=B(1+x)^{-1/2}\sin(\omega ln(1+x))$ satisfying (3) and this boundary condition. Now

$$0=\phi(1)=2^{-1/2}B\sin(\omega ln(2))\ \rightarrow\ \sin(\omega ln(2))=0\ \rightarrow\ \omega ln(2)=n\pi\ ,\ n\geq 1\ .$$

Thus, $\omega^2=\lambda-1/4=(n\pi/ln(2))^2$. Therefore, the eigenvalues and associated eigenfunctions for this problem are

$$\begin{cases} \lambda_n = 1/4+(\frac{n\pi}{ln(2)})^2 & n=1,2,3,\ldots \\ \phi_n(x)=(1+x)^{-1/2}\sin(\frac{n\pi}{ln(2)}ln(1+x)) & \end{cases} \tag{4}$$

This gives us the solution form for problem (2):

$$u(x,t)=(1+x)^{-1/2}e^{-D_0t/4}\sum_{n=1}^{\infty}B_ne^{-D_0n^2\pi^2t/(ln(x))^2}\sin\left(\frac{n\pi}{ln(2)}ln(1+x)\right)\ . \tag{5}$$

Remark: If we would not have made the above positivity assumption on λ in Example 1, then assume $\lambda<1/4$ and define $\alpha:=\frac{1}{2}\sqrt{1-4\lambda}>0$. Then solution of the characteristic equation would be $r=-1/2\pm\alpha$, and so $\phi(x)=(1+x)^{-1/2}\{A(1+x)^{\alpha}+B(1+x)^{-\alpha}\}$. Now $\phi(0)=0=A+B$, so $\phi(x)=A(1+x)^{-1/2}\{(1+x)^{\alpha}-(1+x)^{-\alpha}\}$, while $\phi(1)=0=A2^{-1/2}[2^{\alpha}-2^{-\alpha}]$, which implies $A=0$ since $\alpha>0$; so $\phi\equiv 0$. Therefore, there is no eigenvalue $\lambda<1/4$. We'll leave it as an exercise to draw the same conclusion about $\lambda=1/4$.

Exercise: A variable density vibrating string problem
Determine the eigenvalue problem for the following problem, and derive the eigenvalues and associated eigenfunctions:

$$\begin{cases} (1+x)^{-2}u_{tt}=c^2u_{xx} & 0<x<l\ ,\ t>0\ ,\ c>0\ \text{ is a constant} \\ u(x,0)=f(x)\ ,\ u_t(x,0)=0 & 0<x<l \\ u(0,t)=0=u(l,t) & \end{cases}$$

Exercise: In considering acoustic measurements in a thin tube scaled to be of unit length, let $p(x,t)$ be acoustic pressure, and $v(x,t)$ be volume velocity. Given certain assumptions, one model to consider is

$$\frac{\partial p}{\partial x} = -\frac{\rho}{A(x)}\frac{\partial v}{\partial t}\,, \quad \frac{\partial v}{\partial x} = -\frac{A(x)}{\rho c^2}\frac{\partial p}{\partial t}\,,$$

where $A(x)$ is the variable cross-sectional area of the tube at location x, ρ is the air density in the tube, and c is the speed of sound. Suppose we have scaled the problem so that $c = 1$ and $\rho = 1$. Assume also that $A(x)$ is a continuously differentiable function, and $A(x) > 0$ on $[0,1]$. First eliminate v in the above system to obtain a single equation for $p(x,t)$. Let $p(0,t) = 0$ and $p_x(1,t) = 0$ for all $t > 0$. Then separate variables, $p(x,t) = T(t)\phi(x)$, to obtain the EVP for this p-problem. (The eigenvalue equation is sometimes called Webster's horn equation.) For general $A(x)$ satisfying the above conditions, if we assume real eigenvalues, then

1. Show that any eigenvalue λ must satisfy $\lambda \geq 0$.

2. Show that $\lambda = 0$ is not an eigenvalue of the problem.

3. In the special case $A(x) = e^{ax}$, where $a \neq 0$, write out what the EVP is in this case. Then show that the eigenvalues λ_n must satisfy the transcendental equation $2\sqrt{\lambda - a^2/4}/a = tan(\sqrt{\lambda - a^2/4})$, and hence, there is an infinite ordered set of them, with $\lambda_n \to \infty$ as $t \to \infty$.

Example 2: ***Symmetric diffusion in a disk***
For multidimensional diffusion equations, special cases arise in the case of domains with nice geometry, for example disk and wedge shaped spatial domains in $\mathbb{R}^2$, and spherical domains in $\mathbb{R}^3$. In the 2D situation, the Laplacian in polar coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{6}$$

or in cylindrical coordinates, we have

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{7}$$

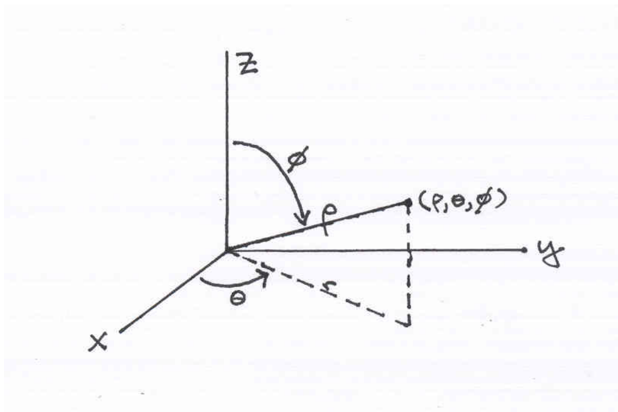

Figure 1: Coordinate angle definitions that will be used for spherical coordinates in these Notes.

and in spherical coordinates (see Figure 1),

$$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{cot\phi}{\rho^2}\frac{\partial}{\partial\phi}\ . \tag{8}$$

In fact, the *radial part of the Laplacian*, in arbitrary n dimensions ($n \geq 1$),with r notationally denoting the radial distance from the origin, is given by

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\ .$$

We will briefly look at some special problems in higher dimensions later in these Notes, but for now let us consider the diffusion equation on the disk spatial domain $\Omega := \{(r,\theta) : 0 \leq r < a,\ 0 \leq \theta < 2\pi\}$, and consider the **symmetric** case (u is independent of angle θ)

$$\begin{cases} u_t = \frac{D}{r}(ru_r)_r & r < a\ ,\ t > 0\ ,\ D > 0 \text{ is a constant} \\ u(r,0) = f(r) & r < a \\ u(a,t) = 0 & u \text{ remains bounded on } \Omega \end{cases} \tag{9}$$

Let $u(r,t) = T(t)\phi(r)$, then $(1/DT)\frac{dT}{dt} = \frac{1}{r\phi}\frac{d}{dr}(r\frac{d\phi}{dr}) = -\lambda$, so $dT/dt = -\lambda DT$, as usual, and

$$\frac{d}{dr}(r\frac{d\phi}{dr}) + \lambda r\phi = 0 = r\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} + \lambda r\phi \tag{10}$$

with $\phi(a) = 0$ and ϕ is bounded at $r = 0$. Note that (10) is **not** a Cauchy-Euler equation, because of the r dependence associated with the λ term. But it is a well-studied equation because it arises so much in practice; (10) is **Bessel's equation of order 0**, and we'll study this variable coefficient EVP later. However, an introduction to Bessel's equation and Bessel functions is given in Appendix F.

The point here in introducing these examples is to motivate us to briefly study a more general class of EVPs called **regular Sturm-Liouville Eigenvalue problems**. They have the form

$$\begin{cases} \frac{d}{dx}(p(x)\frac{d\phi}{dx}) - q(x)\phi + \lambda\sigma(x)\phi = 0 \quad a < x < b \\ \alpha\phi(a) + \beta\frac{d\phi}{dx}(a) = 0 \\ \gamma\phi(b) + \delta\frac{d\phi}{dx}(b) = 0 \end{cases} \tag{11}$$

The functions and parameters in (11) must meet the following conditions:

- $p(x)$ is continuous on $[a, b]$, continuously differentiable on (a, b), $p(x) > 0$ on $[a, b]$.
- $q(x), \sigma(x)$ are continuous on $[a, b]$, $\sigma(x) > 0$, $q(x) \geq 0$ on $[a, b]$.
- $\alpha, \beta, \gamma, \delta$ are real constants.

Remark: The sign convention on the q term in the equation is not universal. We use the negative sign in the equation so all the inequalities in the above conditions are either "$>$" or "$\geq$".

Example 3: In our usual example $\phi'' + \lambda\phi = 0$, $0 < x < l$, $\phi(0) = 0 = \phi(l)$, $p(x) \equiv 1$, $q(x) \equiv 0$, $\sigma(x) \equiv 1$, and $\beta = \delta = 0$. Then we obtain $\lambda = \lambda_n = n^2\pi^2/l^2$, $\phi = \phi_n(x) = \sin(n\pi x/l)$, $n = 1, 2, 3, \ldots$. Given that we have explicit representations for the eigenvalues and eigenfunctions we can make some straightforward observations:

1. The eigenvalues are real and ordered; that is, $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$, with $\lambda_n \to \infty$ as $n \to \infty$.
2. Corresponding to each λ_n is an eigenfunction, $\phi_n = \sin(n\pi x/l)$, that has $n - 1$ zeros in the interval $(0, l)$ (see Figure 2).

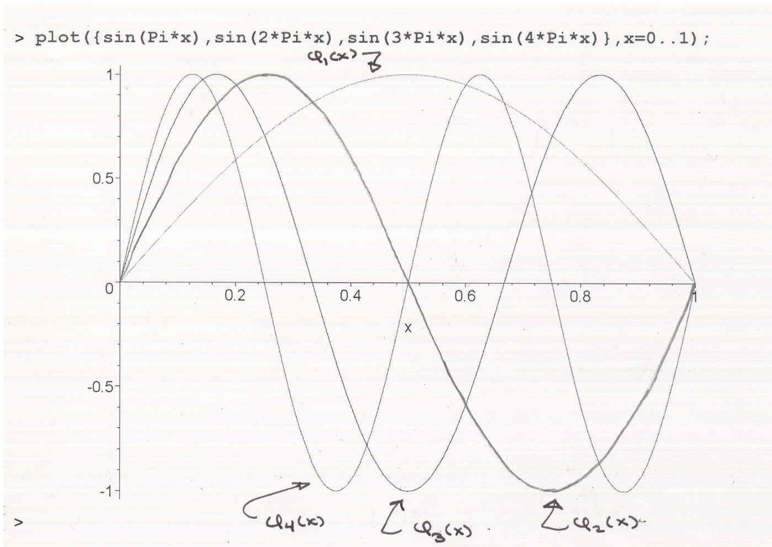

Figure 2: Note the splicing of zeroes of successive eigenfunctions for Example 3.

3. The eigenfunctions $\{\sin(n\pi x/l)\}_{n\geq 1}$ form an orthogonal set of functions on $(0,l)$; that is,

$$< \phi_n, \phi_m >:= \int_0^l \phi_n(x)\phi_m(x)dx =$$

$$\int_0^l \sin(n\pi x/l)\sin(m\pi x/l)dx = \begin{cases} 0 & \text{if } n \neq m \\ l/2 & \text{if } n = m \end{cases} .$$

4. The eigenfunctions are **complete** with respect to the set of piecewise smooth functions f on $(0,l)$; that is, we can, for such a function f, write $f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x)$, where the infinite sum converges for all $x \in (0,l)$, to $[f(x+)+f(x-)]/2$, if the coefficients are chosen to be the Fourier coefficients of f. That is, $a_n =< f, \phi_n > / < \phi_n, \phi_n >$.

The goal here is to present the case that problems of the form (11) with coefficients satisfying the bulleted items have the same properties as our prototypical EVP we have been working with. (So our prototypical EVP is a regular Sturm-Liouville Eigenvalue problem.)

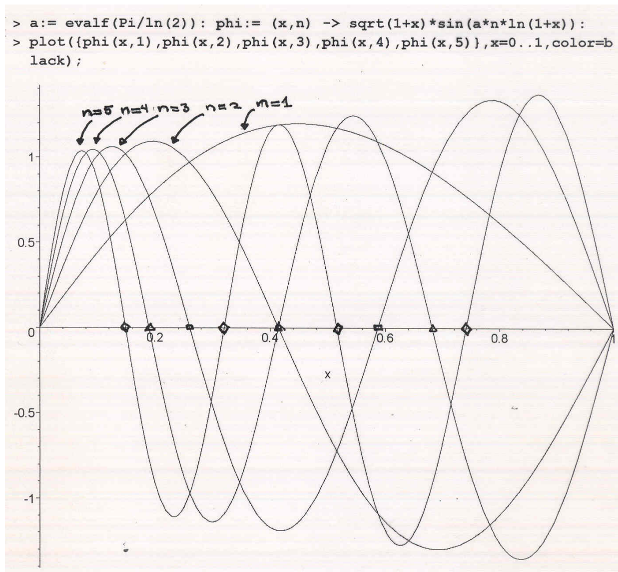

Figure 3: This shows the first 5 eigenfunctions associated with the eigenvalue problem $(1+x)^2\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(1) = 0$.

Example 1, again: Returning to example 1, $p(x) = (1+x)^2$, $q(x) \equiv 0$, and $\beta = \delta = 0$, so this example leads to a regular Sturm-Liouville EVP.

Example 2, again: From (10), $p(r) = r$, $q(r) \equiv 0$, and $\sigma(r) = r$, and on the interval $[0, a]$, $\alpha = 0$, $\delta = 0$. Now the smoothness conditions in the bulleted conditions is satisfied by this problem, but $p(r)$ and $\sigma(r)$ are not strictly positive on the closed interval $[0, a]$. However, p, σ are zero only at the boundary point $r = 0$, otherwise the conditions are met. So example 2 is an example of a **singular** Sturm-Liouville EVP, but it is close enough to the regular case that what properties are brought up below for the regular Sturm-Liouville EVP will also hold the singular Sturm-Liouville EVP too.

Exercise: For the exercise on page 2, what is the p, q, σ for the derived EVP?

Sturm-Liouville Theorem: The regular Sturm-Liouville EVP defined by (11) and the bulleted points below (11) satisfies

1. There exists an infinite number of discrete eigenvalues, λ_n, $n = 1, 2, \ldots$, that are real, positive, ordered, and $\lambda_n \to \infty$ as $n \to \infty$.

2. The eigenfunctions corresponding to different eigenvalues are orthogonal on $[a, b]$ *with respect to* σ; that is, for the eigenvalue-eigenfunction pairs $\{\lambda_i, \phi_i\}, \{\lambda_j, \phi_j\}$, $\lambda_i \neq \lambda_j$,
$< \phi_i, \phi_j >= \int_a^b \phi_i(x)\phi_j(x)\sigma(x)dx = 0$.

3. Eigenfunctions of the same eigenvalue are unique up to multiplicative constant.

4. The nth eigenfunction $\phi_n(x)$ associated with the nth eigenvalue λ_n has exactly $n - 1$ zeros in (a, b). (For an example, see Figure 3.)

5. $\{\phi_n\}_{n\geq 1}$ are complete with respect to piecewise smooth functions f on $[a, b]$. Thus,
$\int_a^b \{f(x) - \sum_1^N a_n\phi_n(x)\}^2\sigma(x)dx \to 0$ as $N \to \infty$.

Our intention is not go through a full proof of this theorem here, but to go through some parts of it to illustrate the arguments. Consult a more advanced treatment of Sturm-Liouville EVPs to get the full story.

For purposes here let the boundary conditions for (11) be the special Dirichlet conditions: $\phi(a) = 0 = \phi(b)$.

Claim 1: Any eigenvalue of (11) is real
Let λ be any eigenvalue of (11), with associate eigenfunction $\phi(x)$. If λ is complex, then $\lambda = \lambda_r + i\lambda_i$ and its complex conjugate is $\bar{\lambda} = \lambda_r - i\lambda_i$, with associated eigenfunction $\psi(x) = \bar{\phi}(x)$. Since $\{\lambda, \phi\}$ satisfies

$$\frac{d}{dx}(p\frac{d\phi}{dx}) - q\phi + \lambda\sigma\phi = 0 \quad \text{on } a < x < b \tag{12}$$
$$\phi(a) = 0 = \phi(b)$$

then, by taking the complex conjugation of the equation (12), and noting that p, q and σ are real functions, $\{\bar{\lambda}, \psi\}$ satisfies

$$\frac{d}{dx}(p\frac{d\psi}{dx}) - q\psi + \bar{\lambda}\sigma\psi = 0 \quad \text{on } a < x < b \tag{13}$$
$$\psi(a) = 0 = \psi(b)$$

So, multiply equation (12) by ψ and multiply (13) by ϕ, then subtract the two resulting equations. This gives us

$$\psi(p\phi')' - \phi(p\psi')' + (\lambda - \bar{\lambda})\sigma\psi\phi = 0 \ .$$

Now integrate:

$$\int_a^b [\psi(p\phi')' - \phi(p\psi')']dx + (\lambda - \bar{\lambda}) \int_a^b \sigma\phi\psi dx = 0 \ \ .$$

By integration-by-parts,

$$\int_a^b [\psi(p\phi')' - \phi(p\psi')']dx = \psi p\phi'|_a^b - \int_a^b p\psi'\phi' dx - \{\phi p\psi'|_a^b - \int_a^b p\phi'\psi' dx\} = 0$$

then

$$(\lambda - \bar{\lambda}) \int_a^b \sigma\phi\psi dx = (\lambda - \bar{\lambda}) \int_a^b |\phi|^2 \sigma dx = 0 \ \ .$$

Since $|\phi|^2 = \phi\psi = \phi\bar{\phi} > 0$, then $\lambda = \bar{\lambda}$, which implies λ is real.

Claim 2: $\lambda > 0$

Let $\{\lambda, \phi\}$ be any eigenvalue-eigenfunction pair, then by (12), $\phi(p\phi')' - q\phi^2 + \lambda\sigma\phi^2 = 0$ on interval (a, b). So,

$$0 = \int_a^b \phi(p\phi')' dx - \int_a^b q\phi^2 dx + \lambda \int_a^b \sigma\phi^2 dx \ .$$

By integration-by-parts, the first integral, after applying the boundary conditions, is $-\int_a^b p(\phi')^2 dx$. Thus,

$$\lambda = \frac{\int_a^b \{p(\phi')^2 + q\phi^2\} dx}{\int_a^b \phi^2 \sigma dx} \geq 0 \ \ . \tag{14}$$

Because of the positivity conditions we imposed on p, q, σ, and the fact the ϕ is a non-zero function, the nominator in (14) is positive, so $\lambda > 0$.

Remark: The right side quotient of (14) can be considered a *functional* of ϕ, so write

$$\lambda = \mathcal{R}[\phi] \ \ .$$

$\mathcal{R}[\cdot]$ is called the **Rayleigh quotient**, and plays a big part in characterizing the eigenvalues in Sturm-Liouville EVPs. An outline of the use of the Rayleigh quotient in characterizing eigenvalues through a minimization principle is presented in Appendix E.

Claim 3: Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to $\sigma(x)$.
Let $\{\lambda, \phi\}, \{\mu, \psi\}$ be two arbitrary eigenvalue-eigenfunction pairs as solutions to (12), with $\lambda \neq \mu$. Thus,

$$(p\phi')' - q\phi + \lambda\sigma\phi = 0\ , \quad \phi(a) = \phi(b) = 0$$

$$(p\psi')' - q\psi + \mu\sigma\psi = 0\ , \quad \psi(a) = \psi(b) = 0$$

Multiply the first equation by ψ, the second equation by ϕ, subtract and integrate:

$$\int_a^b [\psi(p\phi')' - \phi(p\psi')']dx + (\lambda - \mu)\int_a^b \sigma\phi\psi dx = 0\ \ .$$

The first integral is 0 via integration-by-parts and boundary conditions. Since $\lambda \neq \mu$, then $\int_a^b \sigma\phi\psi dx = 0$, which was to be proved.

Claim 4: Eigenfunctions of the same eigenvalue are unique up to a multiplicative constant.
Let ϕ, ψ be two eigenfunctions associated with the same eigenvalue λ. Then

$$\begin{aligned} 0 &= \phi[(p\psi')' - q\psi + \lambda\sigma\psi] - \psi[(p\phi')' - q\phi + \lambda\sigma\phi] \\ &= \phi(p\psi')' - \psi(p\phi')' \\ &= [p(\phi\psi' - \psi\phi')]' \end{aligned}$$

which implies $p(\phi\psi' - \psi\phi') = \text{constant} = C$. Applying the boundary conditions leads to $C = 0$, so

$$\phi\psi' - \psi\phi' = 0\ \ . \tag{15}$$

This statement should be recognizable as the Wronskian of ψ and ϕ. Assume neither ϕ or ψ vanish in the interval, then we can write this expression as $\psi'/\psi = \phi'/\phi$, or $(\ln\psi)' = (\ln\phi)'$, or $\ln\psi - \ln\phi = \text{constant} \quad \rightarrow \quad \ln(\psi/\phi) =$

constant $\quad \rightarrow \psi/\phi =$ constant; that is, $\psi = k\phi$ for some constant k. Now if there is an x_0 where $\phi(x_0) = 0$, for example, then from (15), $\psi(x_0)\phi'(x_0) = 0$. But if $\phi'(x_0) = 0$, then $\phi(x) \equiv 0$ because $\phi(x)$ is the solution to a homogeneous second-order linear ode with zero initial conditions. Since ϕ is an eigenfunction, this can not be the case, so $\psi(x_0) = 0$, which means $\psi = k\phi$ holds automatically at that point.

We will not pursue proving further conclusions of the Sturm-Liouville theorem, but you can see the pattern of reasoning behind it. A point here is that in most cases involving variable coefficient EVPs, we do not have much hope of obtaining an explicit formulas for the eigenvalues and eigenfunctions, but the *general Sturm-Liouville problems behave qualitatively exactly like our simpler, constant coefficient EVP.*

Remark: About an infinite number of eigenvalues going off to infinity: consider the Sturm-Liouville problem (11) again, and write

$$\begin{cases} (p(x)\phi')' - q(x)\phi = -F(x) \quad a < x < b \\ \\ \phi(a) = 0 = \phi(b) \end{cases} \tag{16}$$

where now we forget for a moment that $F(x) = \lambda\sigma(x)\phi(x)$. It turns out, as we discuss later, that there exists a function $G(x, \xi)$, called the **Green's function for the problem** (16), such that the solution to the problem can be written as

$$\phi(x) = \int_a^b G(x, \xi)F(\xi)d\xi \quad .$$

For our eigenvalue problem, F is in terms of the solution (and its eigenvalue), so this statement gives the *integral equation*

$$\phi(x) = \lambda \int_a^b G(x, \xi)\sigma(\xi)\phi(\xi)d\xi \quad . \tag{17}$$

That is, given λ, its associated eigenfunction ϕ satisfies (17), a *Fredholm integral equation of the first kind*. The study of integral equations was very intense in the early part of the twentieth century, and has been a valuable way to obtain properties of solutions to ordinary (and partial) differential equations. One of the consequences, when $\lambda = \lambda_n$ and $\phi = \phi_n(x)$ is a

Bessel's inequality

$$\sum_{n=1}^{\infty} \frac{\phi_n^2}{\lambda_n^2 \int_a^b \sigma \phi_n^2 dx} \le \int_a^b G(x,\xi)^2 \sigma(\xi) d\xi \quad ,$$

which gives

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \int_a^b \sigma(x) \sum_{n=1}^{\infty} \frac{\phi_n^2(x)}{\lambda_n^2 \int_a^b \sigma \phi_n^2 d\xi} dx \le \int_a^b \int_a^b G(x,\xi)^2 \sigma(\xi)\sigma(x) d\xi dx < \infty \ ,$$

so $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is a convergent series. This implies $1/\lambda_n^2 \to 0$ as $n \to \infty$; that is, $\lambda_n \to \infty$ as $n \to \infty$.

Summary: Make sure you know the definition of a regular Sturm-Liouville EVP, and in our limited discussion what distinguishes it from a *singular* Sturm-Liouville EVP. You should know the statement of the Sturm-Liouville Theorem, that is, the properties of the solutions $\{\lambda_n, \phi_n\}$. Finally, be able to recall the Rayleigh quotient for a given problem.

Exercises: Consider the eigenvalue problem

$$\begin{cases} \frac{d^2\phi}{dx^2} - \nu \frac{d\phi}{dx} + \lambda \phi = 0 & 0 < x < \pi \\ \\ \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(\pi) = 0 & \nu > 0 \text{ is a constant} \end{cases}$$

1. Put the equation into Sturm-Liouville form. What functions correspond to $p(x), q(x), \sigma(x)$? From this what do you know about the eigenvalues and eigenfunctions without trying to compute them?

2. Derive the set of eigenvalues and associated eigenfunctions for this EVP.

3. In the next section we will mention PDE eigenvalue problems, but a non-standard one is the **Stekloff problem**

$$\begin{array}{ll} \nabla^2 u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega \end{array}$$

where the eigenvalue appears in the boundary condition[1]. Consider a 1D problem with $\Omega = (0, 1)$, so the equation becomes $u'' = 0$ in $(0, 1)$. Determine the set of eigenvalues for this problem.

[1] ν is the unit vector defined on the boundary of Ω, so $\partial u/\partial \nu$ is the flux of u out of the domain.